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ON LOWER BOUND TREES FOR THE RADIO NUMBER

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Abstract: A radio labeling of a graph G is a function f from the set of vertices $V(G)$ to the set of non-negative integers such that $|f(u) - f(v)| \geq \text{diam}(G) + 1 - d(u, v)$ for every pair of distinct vertices u, v of G. The radio number of G, denoted by $\text{rn}(G)$, is the smallest number k such that G has radio labeling f with $\max\{f(v) : v \in V(G)\} = k$. In [11, Theorem 3], Liu gave a lower bound for the radio number of trees and presented a class of trees, namely spiders, achieving the lower bound. A tree T is called a lower bound tree for the radio number if the radio number of T is equal to the lower bound given in [11, Theorem 3]. In this paper, we give two techniques which convert any tree to lower bound tree for the radio number by adding new vertices to given tree.

Keywords and Phrases: Radio labeling, radio number, tree.

2020 Mathematics Subject Classification: 05C12, 05C15, 05C78.

1. Introduction

In a telecommunication system, the interference constraints between two transmitters play a very important role to design radio networks. The channels are assigned to the transmitters with least use of spectrum such that all the interference constraints are fulfilled for the radio network which is known as optimal channel assignment. The level of interference between two transmitters is closely related to the locations between them. The closer the location higher the interference is. In order to avoid the interference, the gap between the channels assigned to transmitters should be larger. Two transmitters in the network are known as very close transmitters if the interference between them is highest.

This optimal channel assignment problem is also studied using graph model in which transmitters are represented by vertices and two vertices are adjacent if the corresponding transmitters are very close. A *radio labeling* of a graph G is a mapping f from the set of vertices to the set of non-negative integers such that the following holds for any two distinct vertices $v_1, v_2 \in V(G)$:

$$|f(v_1) - f(v_2)| \geq \text{diam}(G) + 1 - d(v_1, v_2).$$

The span of f, denoted by span(f), is defined as span$(f) = \max\{|f(v_1) - f(v_2)| : v_1, v_2 \in V(G)\}$. The *radio number* of G is defined as

$$\text{rn}(G) = \min\{\text{span}(f) : \ f \text{ is a radio labeling of } G\}.$$

Observe that any optimal radio labeling always assign 0 to some vertex. Since $d(v_1, v_2) \leq \text{diam}(G)$, note that a radio labeling is a one-to-one integral function from the set of vertices to the set of non-negative integers. Hence a radio labeling f induces a linear order $u_0, u_1, \ldots, u_{n-1}$ of vertices of G such that

$$0 = f(u_0) < f(u_1) < \ldots < f(u_{n-1}) = span(f).$$

A linear order $u_0, u_1, \ldots, u_{n-1}$ is called an optimal linear order if it is induced by some optimal radio labeling f. We denote an optimal linear order $u_0, u_1, \ldots, u_{n-1}$ of $V(G)$ by $\vec{u}$. It is clear that an optimal linear order $\vec{u}$ of $V(G)$ is not unique. In fact, if $u_i, 0 \leq i \leq n-1$ is an optimal linear order then a linear order $v_i = u_{n-1-i}$ is also an optimal linear order.

A radio labeling problem is one of the hard graph labeling problems. The determination of radio number for any graph is a tough task and it is known for handful graph families only. A brief survey on the radio number of graphs is published by Chartrand and Zhang in [4]. The radio number of trees attracted many researchers. The radio number of paths was studied by Chartrand *et al.* in [6] which was continued by Liu and Zhu in [12] and they determined the exact radio number for it. In [11], Liu gave a lower bound for the radio number of trees and presented a class of trees namely spiders achieving this lower bound. In [1, 2], Bantva *et al.* gave a lower bound which is same as one given by Liu in [11] but using different notations, and presented a necessary and sufficient condition to achieve this lower bound. Using these results, they determined the radio number for three classes of trees namely banana trees, firecrackers trees and a special class of caterpillars achieving this lower bound. In [10], Li *et al.* determined the radio number of complete m-ary trees while in [8], Halász and Tuza determined the radio number of level-wise regular trees.

Denote the lower bound for the radio number of trees given in [11, Theorem 3] by $lb(T)$. A tree T is called lower bound tree for the radio number if $\text{rn}(T) = lb(T)$ and non-lower bound tree otherwise. In this paper, we give two techniques which convert any tree to a lower bound tree. The techniques are also useful to create large lower bound trees by applying repeatedly. We illustrate both the techniques with examples.

2. Preliminaries

In this section, we define terms and notation which are necessary for the present work. We also recall some known results which will be used in the present work. We follow [13] for standard graph theoretic definitions and notations. The *distance* $d_G(u, v)$ between two vertices u and v is the length of a shortest path joining u and v in G. The *diameter* of a graph G is $\max\{d_G(u, v) : u, v \in V(G)\}$. A *tree* T is a connected acyclic graph. For a tree T, denote *vertex set* and *edge set* by $V(T)$ and $E(T)$. In [11], the weight of T from $v \in V(T)$ is defined as $w_T(v) = \sum_{u\in V(T)} d_T(u, v)$ and the weight of T as $w(T) = \min\{w_T(v) : v \in V(T)\}$. A vertex $v \in V(T)$ is a weight center of T if $w_T(v) = w(T)$. Denote the set of weight center(s) by $W(T)$. In [11], it is proved that the following hold for $W(T)$.

Lemma 2.1. [11] *If w is a weight center of a tree T. Then each component of $T - w$ contains at most $|V(T)|/2$ vertices.*

Lemma 2.2. [11] *Every tree T has one or two weight centers, and T has two weight centers, say, $W(T) = \{w, w'\}$ if and only if ww' is an edge of T and $T - ww'$ consists of two equal sized components.*

In [11], the author viewed a tree T rooted at a weight center w and defined the *level function* on $V(T)$ from fix root w by $L_w(u) = d_T(w, u)$ for any $u \in V(T)$. For any two vertices u and v, if u is on the (w, v)-path (w is a weight center), then u is an *ancestor* of v, and v is *descendent* of u. If u be a neighbour of a weight center w then the subtree induced by u together with all its descendants is called a *branch* at u. Two branches are called *different* if they are induced by two different vertices adjacent to the same weight center w. Using these terms and notations, Liu gave a lower bound for the radio number of trees and a necessary and sufficient condition to achieve the lower bound as following in [11].

Theorem 2.3. *Let T be an n-vertex tree with diameter d. Then*

$$\mathrm{rn}(T) \geq (n-1)(d+1) + 1 - 2w(T). \tag{2.1}$$

Moreover, the equality holds if and only if for every weight center w^*, there exist a radio labeling f with $0 = f(u_0) < f(u_1) < \ldots < f(u_{n-1})$, where all the following hold (for all $0 \leq i \leq n-2$);

(a) *u_i and u_{i+1} are in different branches (unless one of them is w^*);*

(b) *$\{u_0, u_{n-1}\} = \{w^*, v\}$, where v is some vertex with $L_{w^*}(v) = 1$;*

(c) *$f(u_{i+1}) = f(u_i) + d + 1 - L_{w^*}(u_i) - L_{w^*}(u_{i+1})$.*

A tree T for which $\mathrm{rn}(T)$ is given by the right-hand side of (2.1) is called a *lower bound tree* and non-lower bound tree otherwise. Some known lower bound trees as well as non-lower bound trees are as follows. The paths P_{2k} are lower bound trees while P_{2k+1} are not lower bound trees. The banana trees and firecrackers trees are lower bound trees whose radio number is given in [2]. The complete m-ary trees whose radio number is determined in [10] are lower bound trees if $m \geq 3$ and non-lower bound trees if $m = 2$. The level-wise regular trees when all internal vertices have degree more than two which are presented in [8] are lower bound trees. Note that even an addition or a deletion of a vertex or an edge make lower bound tree to non-lower bound tree and vice-versa. For example, it is known that a path P_{2k+1} ($k \geq 1$) is not a lower bound tree but deletion of one leaf vertex from a path P_{2k+1} makes a path P_{2k} which is a lower bound tree. Similarly, the converse procedure of above for a path P_{2k} makes a lower bound tree to non lower bound tree.

Further recall that a tree represents a network of transmitters and it is expected such a tree is a lower bound tree as the lower bound tree minimize the spectrum of channels. But it is not always possible that given tree is a lower bound tree and then the following natural question arise.

Question 2.4. *Is it possible to convert any non-lower bound tree T to a lower bound tree T' by adding new vertices to T ?*

The answer of the above question is affirmative. We give two techniques which convert any tree to a lower bound tree by adding some extra vertices. These techniques can also be used to known lower bound tree to generate large lower bound trees. As a result, the techniques can be used indefinitely to generate more lower bound trees.

3. Main Results

In this section, we continue to use the terminology and notation defined in the previous section. We present some techniques which convert any tree T to the lower bound tree by adding new vertices at center with some conditions.

Let T be any tree of order n and diameter $d \geq 3$. We view a tree T rooted at its $C(T)$. That is, if $C(T) = \{w\}$ then T is rooted at w and if $C(T) = \{w, w'\}$ then T is rooted at both w and w' (both are at level 0). Let T_u be the branch induced by a vertex u adjacent to $w \in C(T)$. Denote the children of u by $u_0, u_1, \ldots, u_{d(u)-1}$. Inductively, denote the children of $u_{i_1 i_2 \ldots i_t}(i_1, i_2, \ldots, i_t \in \mathbb{N} \cup \{0\}, 1 \leq t \leq d/2 - 2)$ by $u_{i_1 i_2 \ldots i_t i_{t+1}}(i_{t+1} \in \mathbb{N} \cup \{0\})$. We find the path partition of a branch T_u of T as follows: Choose u as a one end vertex and find a longest path in T_u, say this path is P_{m_1}, where $0 \leq m_1 \leq d/2$. Take $V(T_u) \setminus V(P_{m_1})$. Note that the resulting graph is a forest. Let $u_{i_1, i_2, \ldots, i_s} \in V(T_u) \setminus V(P_{m_1})$, where $1 \leq s \leq d/2 - 1$ be the closest descendent vertex of u in T_u then choose $u_{i_1, i_2, \ldots, i_s}$ as a one end vertex and find longest path in $T_u \setminus P_{m_1}$, say this path is P_{m_2}. Continue this process until the remaining forest has vertex of degree three. The single vertex is considered as P_1 in this operation. Let $T_1, T_2, \ldots, T_k$ be the branches of $T - w$ when $C(T) = \{w\}$. Then partition each branch $T_i, 1 \leq i \leq k$ as describe above which we called the path partition of tree T rooted at w. Let $C_w = \{n_1 P_1, n_2 P_2, \ldots, n_{d/2} P_{d/2}\}$ be the collection of all paths obtained by path partition of T rooted at w, where $n_i P_i, 1 \leq i \leq d/2$ denote the n_i copies of path P_i. Let T_w and $T_{w'}$ be the components of $T - ww'$ when $C(T) = \{w, w'\}$. Note that both T_w and $T_{w'}$ are trees. Now we view T_w and $T_{w'}$ rooted at w and w', respectively and find its path partition as described above. Let $C_w = \{n_1 P_1, n_2 P_2, \ldots, n_{\lfloor d/2 \rfloor} P_{\lfloor d/2 \rfloor}\}$ and $C_{w'} = \{n'_1 P_1, n'_2 P_2, \ldots, n'_{\lfloor d/2 \rfloor} P_{\lfloor d/2 \rfloor}\}$ be the collection of all paths obtained by path partition of T_w and $T_{w'}$, respectively. We now construct a tree T_p using a given tree T as follows: (1) If $C(T) = \{w\}$ then join one end vertex of each copy of path $P_i \in C_w$ and one more vertex w_1 by an edge to w; (2) If $C(T) = \{w, w'\}$ then join one end vertex of each copy of path P_i of $C_{w'}$ by an edge to w and each copy of path P_j of C_w by edge to w' and one more vertex to each w and w' by an edge. It is clear that $|T_p| = 2n$ and $\text{diam}(T_p) = \text{diam}(T)$.

Lemma 3.1. *Let T_p be the tree obtained as above, then the following hold for T_p.*

(a) *If $C(T) = \{w\}$ then $w \in W(T_p)$,*

(b) *If $C(T) = \{w, w'\}$ then one of $\{w, w'\}$ or, both w and w' are in $W(T_p)$.*

Proof. We consider the following two cases.

Case - 1. $|C(T)| = 1$.

If possible then assume that $w \notin W(T_p)$. Note that there are two possibilities for $W(T_p)$; (1) $|W(T_p)| = 1$, (2) $|W(T_p)| = 2$. We consider both the cases separately as follows.

Subcase - 1. $|W(T_p)| = 1$.

In this case, let $W(T_p) = \{w_1\}$. Since $w \notin W(T_p)$, it is clear that $w \neq w_1$. Let $T_1, T_2, \dots, T_k$ be the components of $T_p - w_1$. Without loss of generality, assume that $w \in T_1$ then note that $|T_1| \geq n + 1$ which is a contradiction with Lemma 2.1.

Subcase - 2. $|W(T_p)| = 2$.

In this case, let $W(T_p) = \{w_1, w_2\}$. Since $w \notin W(T_p)$, it is clear that $w \neq w_1, w_2$. Let T_1 and T_2 be two component of $T_p - w_1w_2$. Without loss of generality, assume that $w \in T_1$ then note that $|T_1| \geq n + 1$ which is a contradiction with Lemma 2.2.

Thus, in both the subcases above, our assumption $w \notin W(T_p)$ is wrong and hence $w \in W(T_p)$.

Case - 2. $C(T) = \{w, w'\}$.

If possible then assume that $w, w' \notin W(T_p)$. Again note that there are two possibilities for $W(T_p)$; (1) $|W(T_p)| = 1$, (2) $|W(T_p)| = 2$. We consider both the cases separately as follows.

Subcase - 1. $|W(T_p)| = 1$.

In this case, let $W(T_p) = \{w_1\}$. Since $w, w' \notin W(T_p)$, it is clear that $w, w' \neq w_1$. Let $T_1, T_2, \dots, T_k$ be the component of $T_p - w_1$. Without loss of generality, assume that $w \in T_1$ then note that $|T_1| \geq n + 1$ which is a contradiction with Lemma 2.1.

Subcase - 2. $|W(T_p)| = 2$.

In this case, let $W(T_p) = \{w_1, w_2\}$. Since $w, w' \notin W(T_p)$, we have $w, w' \neq w_1, w_2$. Then $T_p - w_1w_2$ has two equal size component, say T_1 and T_2. Since $ww' \neq w_1w_2$, w and w' belongs to the same component of $T_p - w_1w_2$. Without loss of generality, assume $w, w' \in T_1$ then note that $|T_p| \geq n+2$ which is a contradiction with Lemma 2.2.

Thus, in both the subcases above, our assumption $w, w' \notin W(T_p)$ is wrong and hence one of $\{w, w'\}$ or both w and w' are in $W(T_p)$ which completes the proof.

Theorem 3.2. *Let T be any tree of order n and diameter $d \geq 2$, and T_p is a tree obtained from T as above. Then T_p is a lower bound tree and*

$$\mathrm{rn}(T_p) = (2n - 1)(d + 1) - 2w(T_p) + 1. \tag{3.1}$$

Proof. We consider the following two cases.

Case - 1. $|C(T)| = 1$.

Let $C(T) = \{w\}$ then by Lemma 3.1, it is clear that $w \in W(T_p)$. We view a tree T rooted at w. Denote $v_1, v_2, \ldots, v_n$ be the vertices of T and $v'_1, v'_2, \ldots, v'_n$ be the newly added vertices as descendent of w such that $d(v'_i, v_{i+1}) \leq d/2 + 1$ for $1 \leq i \leq n$. By the construction of T_p observe that such an ordering is possible. Define a linear order $u_0, u_1, \ldots, u_{2n-1}$ of $V(T_p)$ as follows: Set $u_0 = v_1, u_{2n-1} = v'_n$ and for $1 \leq t \leq 2n-2$ as follows:

$$u_t = \begin{cases} v'_{(t+1)/2}, & \text{if } t \text{ is odd,} \\ v_{(t/2)+1}, & \text{if } t \text{ is even.} \end{cases}$$

Then note that $u_0 \in W(T), u_{2n-1} \in N(W(T))$ and, u_t and u_{t+1} are in different branches of T_p. Define f by $f(u_0) = 0$ and $f(u_{i+1}) = f(u_i) + d + 1 - L(u_i) - L(u_{i+1})$ for $0 \leq i \leq 2n-2$.

Claim - 1. f is a radio labeling of T_p and $\text{span}(f) = (2n-1)(d+1) - 2w(T_p) + 1$.

Let u_i and u_j be two arbitrary vertices of T_p. If $j - i = 1$ then $f(u_j) - f(u_i) = f(u_{i+1}) - f(u_i) = d + 1 - L(u_i) - L(u_{i+1}) \geq d + 1 - d(u_i, u_{i+1}) = d + 1 - d(u_i, u_j)$. If $j - i \geq 2$ then $f(u_j) - f(u_i) \geq 2(d+1) - d(u_i, u_{i+1}) - d(u_{i+1}, u_j) = 2(d+1) - 2(d/2+1) \geq d \geq d + 1 - d(u_i, u_j)$ as $d(u_i, u_j) \geq 1$. Hence f is a radio labeling of T_p and the span of f is

$$\begin{aligned} \text{span}(f) &= f(u_{2n-1}) - f(u_0) \\ &= \sum_{t=0}^{2n-2} f(u_{t+1}) - f(u_t) \\ &= \sum_{t=0}^{2n-2} (d + 1 - L(u_t) - L(u_{t+1})) \\ &= \sum_{t=0}^{2n-2} d + 1 - \sum_{t=0}^{2n-2} (L(u_t) + L(u_{t+1})) \\ &= (2n-1)(d+1) - 2\sum_{t=0}^{2n-2} L(u_t) + L(u_{2n-1}) \\ &= (2n-1)(d+1) - 2w(T_p) + 1. \end{aligned}$$

Case - 2. $|C(T)| = 2$.

Let $C(T) = \{w, w'\}$ then by Lemma 3.1, either one of $\{w, w'\}$ or both w and w' are in $W(T_p)$. Without loss of generality assume that $w \in W(T_p)$. We view a tree rooted at w. Let T_1 and T_2 be two components of $T - ww'$. Denote vertices of T_1 by $w = v_1, v_2, \ldots, v_{|T_1|}$ and vertices of T_2 by $v_{|T_1|+1}, \ldots, v_n = w'$. Denote newly added vertices as descendent to w' by $v'_1, v'_2, \ldots, v'_{|T_1|}$ and newly added vertices as descendent to w by $v'_{|T_1|+1}, \ldots, v'_n$ such that $d(v'_i, v_{i+1}) \leq (d+3)/2$ for $1 \leq i \leq 2n-2$. Note that such an ordering is possible by the construction of T_p. Define a linear order $u_0, u_1, \ldots, u_{2n-1}$ of $V(T_p)$ as follows: Set $u_0 = v_1 = w, u_{2n-1} = v_n = w'$ and for $1 \leq t \leq 2n-2$ as follows: For $1 \leq t \leq 2|T_1|-1$, define

$$u_t = \begin{cases} v'_{(t+1)/2}, & \text{if } t \text{ is odd,} \\ v_{t/2+1}, & \text{if } t \text{ is even.} \end{cases}$$

and for $2|T_1| \leq t \leq 2n-2$, let

$$u_t = \begin{cases} v_{(t+1)/2}, & \text{if } t \text{ is odd,} \\ v'_{t/2+1}, & \text{if } t \text{ is even.} \end{cases}$$

Then note that $u_0 \in W(T_p), u_{2n-1} \in N(u_0)$ and u_t, u_{t+1} are in different branches of T_p. It is clear that $d(u_i, u_{i+1}) \leq (d+3)/2$. Define f by $f(u_0) = 0$ and $f(u_{i+1}) = f(u_i) + d + 1 - L(u_i) - L(u_{i+1})$ for $1 \leq i \leq 2n-2$.

Claim - 2. f is a radio labeling of T_p and span$(f) = (2n-1)(d+1) - 2w(T_p) + 1$.

Let u_i and u_j be two arbitrary vertices of T_p. If $j - i = 1$ then $f(u_j) - f(u_i) = f(u_{i+1}) - f(u_i) = d + 1 - L(u_i) - L(u_{i+1}) \geq d + 1 - d(u_i, u_{i+1}) = d + 1 - d(u_i, u_j)$. If $j - i \geq 2$ then $f(u_j) - f(u_i) \geq 2(d+1) - d(u_i, u_{i+1}) - d(u_{i+1}, u_j) = 2(d+1) - 2((d+3)/2) = d - 1 \geq d + 1 - d(u_i, u_j)$ as $d(u_i, u_j) \geq 2$. Hence f is a radio labeling of T_p and the span of f is

$$\begin{aligned} \text{span}(f) &= f(u_{2n-1}) - f(u_0) \\ &= \sum_{t=0}^{2n-2} f(u_{t+1}) - f(u_t) \\ &= \sum_{t=0}^{2n-2} (d + 1 - L(u_t) - L(u_{t+1})) \\ &= \sum_{t=0}^{2n-2} d + 1 - \sum_{t=0}^{2n-2} (L(u_t) + L(u_{t+1})) \\ &= (2n-1)(d+1) - 2\sum_{t=0}^{2n-2} L(u_t) + L(u_{2n-1}) \\ &= (2n-1)(d+1) - 2w(T_p) + 1. \end{aligned}$$

Example 3.1. In Figure 1, a tree T with $|C(T)| = 1$ (whose path partition is $C_w = \{2P_3, P_2, 2P_1\}$) and the tree T_p obtained from T by above describe procedure are shown along with an ordering of vertices and an optimal radio labeling.

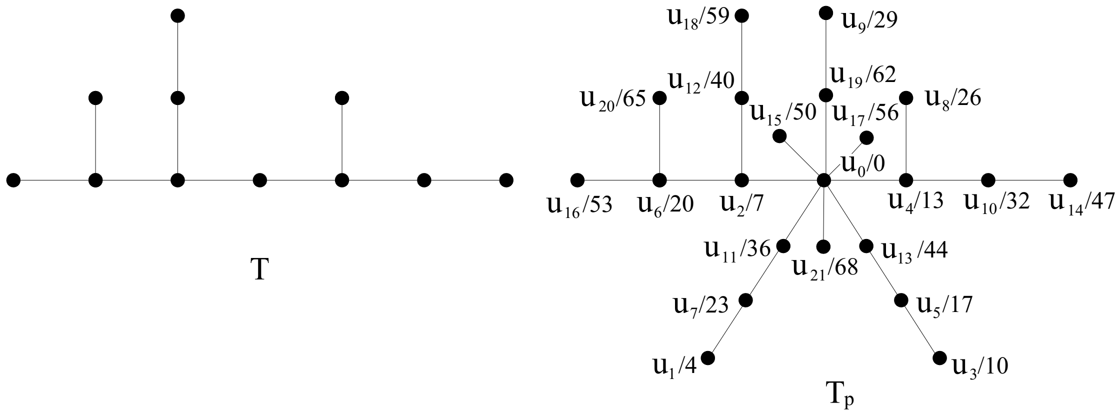

Figure 1: A tree T (left) and its T_p (right) with an optimal radio labeling such that $\mathrm{rn}(T_p) = 68$.

Example 3.2. In Figure 2, the trees T with $|C(T)| = 2$ (whose path partitions are $C_w = \{P_3, 2P_1\}$ and $C_{w'} = \{P_3, P_1\}$) and the trees T_p obtained from T with an ordering of vertices and an optimal radio labeling are shown.

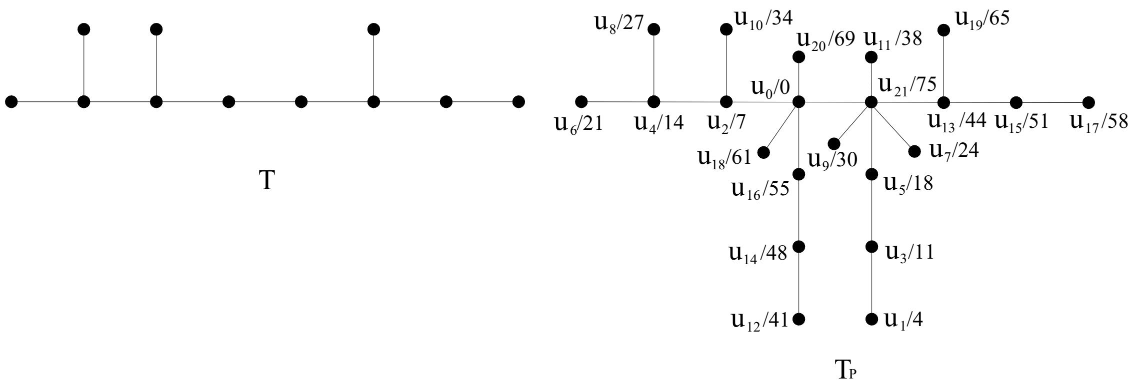

Figure 2: A tree T (left) and its T_p (right) with an optimal radio labeling such that $\mathrm{rn}(T_p) = 75$.

A tree is called a caterpillar if the removal of all its degree-one vertices results in a path, called the spine. Denote the caterpillar with spine vertices $\{v_1, v_2, \ldots, v_n\}$ such that v_i is adjacent to $v_{i+1}, 1 \le i \le n-1$ and $d(v_1) = m_1 + 1, d(v_n) = m_n + 1$ and $d(v_i) = m_i + 2$ for $i = 2, \ldots, n-1$ by $C(m_1, m_2, \ldots, m_n)$. For $1 \le i \le n$, denote the pendant vertices adjacent to v_i by $v_{i,j}$, where $1 \le j \le m_i$ We construct a tree T_c from given tree T of order n and diameter d as follows:

Case - 1. $|C(T)| = 1$. In this case, note that diam(T) is even. Let w be the centre of tree T. Define $V_i = \{v \in V : d(w, v) = i\}$, where $1 \leq i \leq d/2$. Denote by $|V_i| = n_i$. Then it is clear that $\sum_{i=1}^{d/2} n_i + 1 = n$. Take a copy of caterpillar $C(n_{d/2}, n_{d/2} - 1, \ldots, n_2 - 1, n_1)$ and identify the vertex v_1 with w. It is clear that $|T_c| = 2n$ and diameter diam$(T_c) = d =$ diam(T).

Case - 2. $|C(T)| = 2$. In this case, note that diam(T) is odd. Let w and w' be two centres of T. Let T_1 and T_2 be two components of $T - ww'$ such that $w \in T_1$ and $w' \in T_2$. Let $V_i = \{v \in T_1 : d(v, w) = i\}$ and $V_i' = \{v \in T_2 : d(v, w') = i\}$, where $1 \leq i \leq \lfloor d/2 \rfloor$. Denote by $|V_i| = n_i$ and $|V_i'| = n_i'$ for $1 \leq i \leq \lfloor d/2 \rfloor$. Identify the vertex v_1 of a copy of caterpillar $C(n_{\lfloor d/2 \rfloor}, n_{\lfloor d/2 \rfloor} - 1, \ldots, n_2 - 1, n_1)$ with w' and a copy of caterpillar $C(n'_{\lfloor d/2 \rfloor}, n'_{\lfloor d/2 \rfloor} - 1, \ldots, n_2' - 1, n_1')$ with w. It is clear that T_c is a tree with order $2n$ and diameter diam$(T_c) = d =$ diam(T).

Lemma 3.3. *Let T be any tree of order n and diameter $d \geq 2$, and T_c is a tree obtained from T as describe above. Then the following holds.*

(a) *If $C(T) = \{w\}$ then $w \in W(T_c)$.*

(b) *If $C(T) = \{w, w'\}$ then $W(T_c) = \{w, w'\}$.*

Proof. We consider the following two cases.

Case - 1. $C(T) = \{w\}$.

If possible then assume that $w \notin W(T_c)$. We consider the following two cases.

Subcase - 1. $|W(T_c)| = 1$.

Let $W(T_c) = \{w_1\}$. Since $w \neq w_1$, $T - w_1$ has a branch, say T_1 consisting w. It is clear that $|T_1| \geq n + 1$, a contradiction with Lemma 2.1.

Subcase - 2. $|W(T_c)| = 2$.

Let $W(T_c) = \{w_1, w_2\}$. Observe that $w \neq w_1, w_2$ as $w \notin W(T_c)$. Let T_1 and T_2 be two components of $T_c - w_1 w_2$. Since $w \neq w_1, w_2$, without loss of generality, assume that $w \in T_1$ then note that $|T_1| \geq n + 1$ which is a contradiction with Lemma 2.2.

Thus, we have a contradiction in both the above cases and hence, we obtain, $w \in W(T_c)$.

Case - 2. $C(T) = \{w, w'\}$. In this case, it is easy to see that $w, w' \in W(T_c)$; otherwise $T - u$ contains a component with more than $|T_c|/2$ vertices if $u \neq w, w'$ is a weight center of T_c which is a contradiction. Moreover note that ww' is an edge of T_c and by the construction of T_c, it is clear that $w_{T_c}(w) = w_{T_c}(w')$. Hence, by Lemma 2.2, we have $W(T_c) = \{w, w'\}$ which completes the proof.

Theorem 3.4. *Let T be any tree of order n and diameter $d \geq 2$, and T_c is a tree obtained from T as above. Then T_c is a lower bound tree and*

$$\mathrm{rn}(T_c) = (2n-1)(d+1) - 2w(T_c) + 1. \tag{3.2}$$

Proof. We consider the following two cases.

Case - 1. $|C(T)| = 1$.

Let $C(T) = \{w\}$ then by Lemma 3.3, it is clear that $w \in W(T_c)$. We view a tree T rooted at w. Denote the vertices of a tree T by $v_1, v_2, \ldots, v_n$ such that $d/2 = L(v_1) \geq L(v_2) \geq \ldots \geq L(v_n) = 0$. Denote the newly added vertices as $v'_1, v'_2, \ldots, v'_n$ such that $1 = L(v'_1) \leq L(v'_2) \leq \ldots \leq L(v'_n) = d/2$. By the construction of T_c observe that for every $1 \leq k \leq d/2$, $\{v_i : L(v_i) = k\} = \{v'_i : L(v'_i) = d/2 - k + 1\}$. Define a linear order $u_0, u_1, \ldots, u_{2n-1}$ of $V(T_c)$ as follows: Set $u_0 = v_n, u_{2n-1} = v'_1$ and for $1 \leq t \leq 2n-2$, set

$$u_t = \begin{cases} v_{(t+1)/2}, & \text{if } t \text{ is odd,} \\ v'_{(t+2)/2}, & \text{if } t \text{ is even.} \end{cases}$$

Then note that $u_0 \in W(T_c), u_{2n-1} \in N(u_0)$ and, u_t and u_{t+1} are in different branches of T_c. Also note that $d(u_t, u_{t+1}) \leq d/2 + 1$ for $0 \leq t \leq 2n-2$. Define f by $f(u_0) = 0$ and $f(u_{i+1}) = f(u_i) + d + 1 - L(u_i) - L(u_{i+1})$ for $0 \leq i \leq 2n-2$.

Claim - 1. f is a radio labeling of T_c and $\mathrm{span}(f) = (2n-1)(d+1) - 2w(T_c) + 1$.

Let u_i and u_j be two arbitrary vertices of T_c. If $j - i = 1$ then $f(u_j) - f(u_i) = f(u_{i+1}) - f(u_i) = d + 1 - L(u_i) - L(u_{i+1}) \geq d + 1 - d(u_i, u_{i+1}) = d + 1 - d(u_i, u_j)$. If $j - i \geq 2$ then $f(u_j) - f(u_i) \geq 2(d+1) - d(u_i, u_{i+1}) - d(u_{i+1}, u_j) = 2(d+1) - 2(d/2+1) \geq d \geq d + 1 - d(u_i, u_j)$ as $d(u_i, u_j) \geq 1$. Hence f is a radio labeling of T_c and the span of f is

$$\begin{aligned} \mathrm{span}(f) &= f(u_{2n-1}) - f(u_0) \\ &= \sum_{t=0}^{2n-2} f(u_{t+1}) - f(u_t) \\ &= \sum_{t=0}^{2n-2} (d + 1 - L(u_t) - L(u_{t+1})) \\ &= (2n-1)(d+1) - 2\sum_{t=1}^{2n-2} L(u_t) + L(u_{2n-1}) \\ &= (2n-1)(d+1) - 2w(T_c) + 1. \end{aligned}$$

Case - 2. $|C(T)| = 2$.

In this case, denote $C(T) = \{w, w'\}$. Let T_1 and T_2 be two components of $T - ww'$. Without loss of generality assume that $w \in T_1$ and $w' \in T_2$. Denote the vertices of T_1 as $w = v_1, v_2, \ldots, v_{|T_1|}$ such that $0 = L(v_1) < L(v_2) \leq \ldots \leq L(v_{|T_1|}) = \lfloor d/2 \rfloor$ and the vertices of T_2 as $v_{|T_1|+1}, \ldots, v_n = w'$ such that $1 = L(v_{|T_1|+1}) \leq L(v_{|T_1|+2}) \leq \ldots \leq L(v_{n-1}) = \lfloor d/2 \rfloor$. Denote the newly added vertices of caterpillar adjacent to w' by $v'_1, v'_2, \ldots, v'_{|T_1|}$ such that $\lfloor d/2 \rfloor = L(v'_1) \geq L(v'_2) \geq \ldots \geq L(v'_{|T_1|}) = 1$ and the newly added vertices of caterpillar adjacent to w by $v'_{|T_1|+1}, \ldots, v'_n$ such that $\lfloor d/2 \rfloor = L(v'_{|T_1|+1}) \geq \ldots \geq L(v'_n) = 1$.

Define a linear order $u_0, u_1, \ldots, u_{2n-1}$ of $V(T_c)$ as follows: Set $u_0 = v_1 = w$, $u_{2n-1} = v_n = w'$ and for $1 \leq t \leq 2n-2$ as follows: For $1 \leq t \leq 2|T_1| - 1$, define

$$u_t = \begin{cases} v'_{(t+1)/2}, & \text{if } t \text{ is odd,} \\ v_{(t+2)/2}, & \text{if } t \text{ is even.} \end{cases}$$

and for $2|T_1| \leq t \leq 2n-2$, let

$$u_t = \begin{cases} v_{(t+1)/2}, & \text{if } t \text{ is odd,} \\ v'_{(t+2)/2}, & \text{if } t \text{ is even.} \end{cases}$$

Then note that $u_0, u_{2n-1} \in W(T_c)$ and, u_t and u_{t+1} are in different branches for $0 \leq t \leq 2n-2$. Also note that $d(u_t, u_{t+1}) \leq (d+3)/2$. Define f by $f(u_0) = 0$ and $f(u_{i+1}) = f(u_i) + d + 1 - L(u_i) - L(u_{i+1})$ for $0 \leq i \leq 2n-2$.

Claim - 2. f is a radio labeling of T_c and $\text{span}(f) = (2n-1)(d+1) - 2w(T_c) + 1$. Let u_i and u_j be two arbitrary vertices of T_c. If $j - i = 1$ then $f(u_j) - f(u_i) = f(u_{i+1}) - f(u_i) = d + 1 - L(u_i) - L(u_{i+1}) \geq d + 1 - d(u_i, u_{i+1}) = d + 1 - d(u_i, u_j)$. If $j - i \geq 2$ then $f(u_j) - f(u_i) \geq 2(d+1) - d(u_i, u_{i+1}) - d(u_{i+1}, u_j) = 2(d+1) - 2((d+3)/2) = d - 1 \geq d + 1 - d(u_i, u_j)$ as $d(u_i, u_j) \geq 2$. Hence f is a radio labeling of T_c and the span of f is

$$\begin{aligned} \text{span}(f) &= f(u_{2n-1}) - f(u_0) \\ &= \sum_{t=0}^{2n-2} f(u_{t+1}) - f(u_t) \\ &= \sum_{t=0}^{2n-2} (d + 1 - L(u_t) - L(u_{t+1})) \\ &= (2n-1)(d+1) - 2\sum_{t=1}^{2n-2} L(u_t) + L(u_{2n-1}) \\ &= (2n-1)(d+1) - 2w(T_c) + 1. \end{aligned}$$

Example 3.3. In Figure 3, a trees T with $|C(T)| = 1$ and the trees T_c obtained from T with an ordering of vertices and an optimal radio labeling are shown.

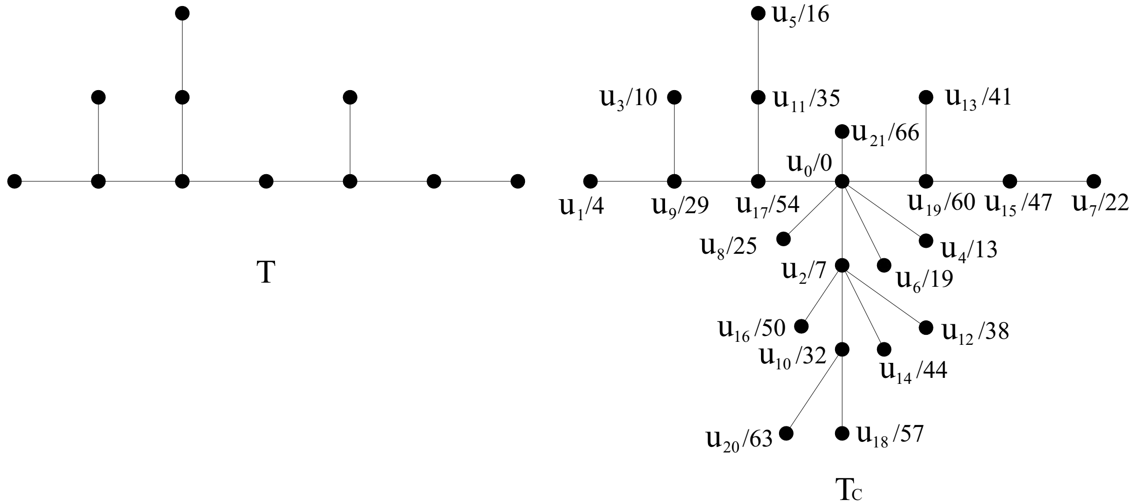

Figure 3: A tree T with $|C(T)| = 1$ and its T_c with $\mathrm{rn}(T_c) = 66$.

Example 3.4. In Figure 4, a trees T with $|C(T)| = 2$ and the trees T_c obtained from T with an ordering of vertices and an optimal radio labeling are shown.

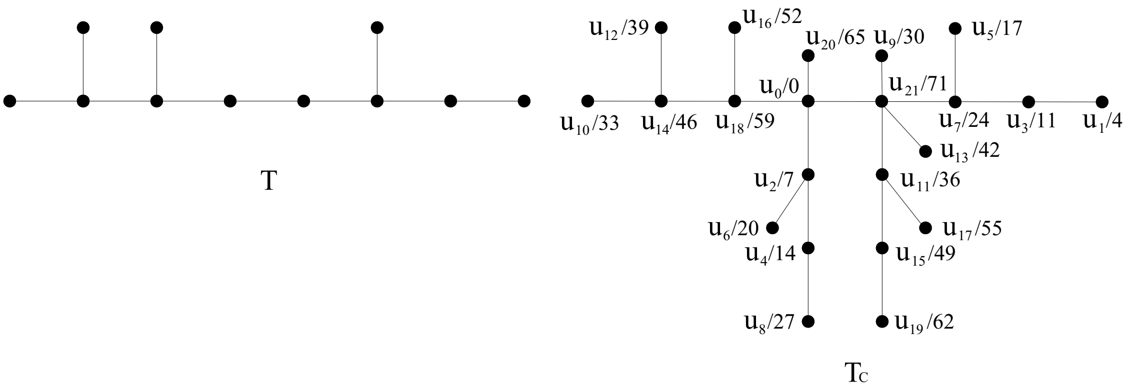

Figure 4: A tree T with $|C(T)| = 2$ and its T_c with $\mathrm{rn}(T_c) = 71$.

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